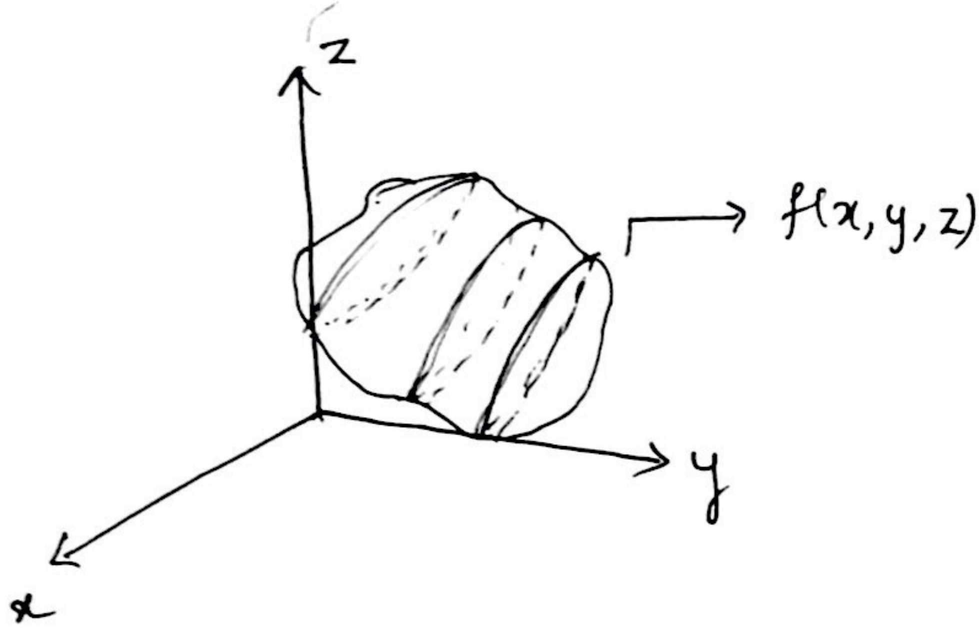
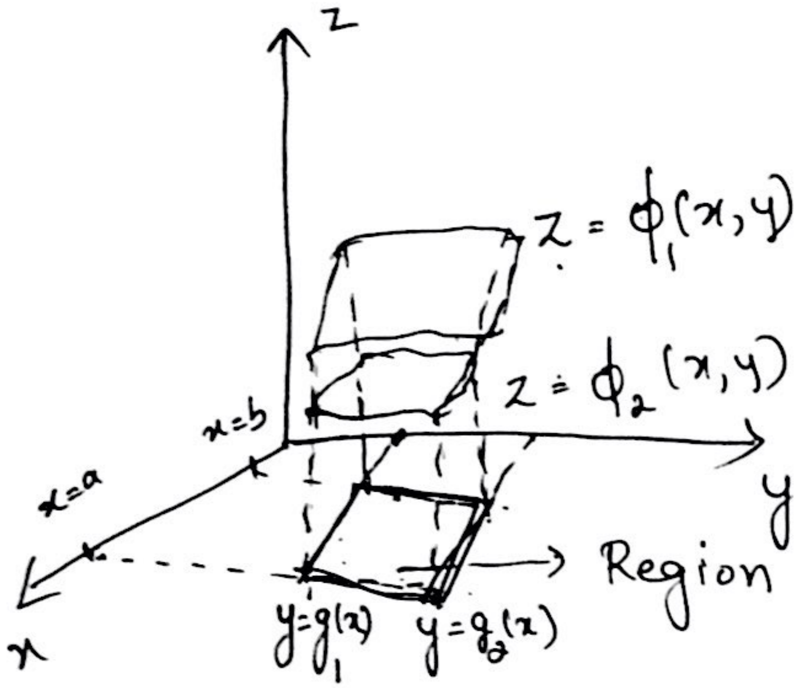
Triple Integral.



Sub division of $f(x,y,z)$ will be very small cubes.



Triple Integrals:

The treatment of Triple Integrals also known as Volume integrals in R^3 is a simple and straight extension of the ideas in respect of double integrals.

Let $f(x,y,z)$ be a continuous and single valued function defined over a region V of space. Let V be divided into sub regions $\delta V_1, \delta V_2, \ldots, \delta V_n$ into n parts. Let (x_k, y_k, z_k) be any arbitrary point within or on the boundary of the sub region δV_k. Form the sum,

$$S = \sum_{k=1}^{n} f(x_k, y_k, z_k)\, \delta V_k \longrightarrow (1).$$

If as $n \to \infty$ and the maximum diameter of every sub-region approaches zero, the sum (1) has a limit, then the limit is denoted by $\iiint_V f(x,y,z)\, dV$. This is called the triple integral of $f(x,y,z)$ over the region V.

For the purpose of evaluation, the above triple integral over the region V can be expressed as an iterated Integral or repeated Integral in the form.

$$\iiint_V f(x,y,z)\,dx\,dy\,dz = \int_a^b \left[\int_{g(x)}^{h(x)} \left\{ \int_{\psi(x,y)}^{\phi(x,y)} f(x,y,z)\,dz \right\} dy \right] dx$$

where $f(x,y,z)$ is continuous in a region V, bounded by surfaces $z=\psi(x,y)$, $z=\phi(x,y)$; $y=g(x)$, $y=h(x)$; $x=a$, $x=b$. The above integral indicates three successive integrations to be performed in the following order, first w.r.t z, keeping x and y as constants, then w.r.t. y keeping x as constant. and finally w.r.t. x.

Note: 1) When an integration is performed w.r.t a variable that Variable is eliminated completely from the remaining integral.

2) If the limits are not constants the integration should be in the order in which dx, dy, dz is given in the integral.

3) Evaluation of the integral may be performed in any order if all the limits are constants.

4) If $f(x,y,z) = 1$ then the triple integral gives the volume of the region.

Problems:

1) Evaluate $\int_0^1 \int_0^2 \int_1^2 xyz^2 \, dx \, dy \, dz$

Solution: $\int_0^1 \int_0^2 \int_1^2 xyz^2 \, dx \, dy \, dz$

$$= \int_0^1 \int_0^2 \left[\frac{x^2}{2} yz^2 \right]_1^2 dy \, dz = \int_0^1 \int_0^2 \left[\frac{(2)^2}{2} yz^2 - \frac{1^2}{2} yz^2 \right] dy dz$$

$$= \int_0^1 \int_0^2 \left[2yz^2 - \tfrac{1}{2} yz^2 \right] dy dz = \int_0^1 \int_0^2 \tfrac{3}{2} yz^2 \, dy dz$$

$$= \int_0^1 \left[\tfrac{3}{2} \tfrac{y^2}{2} z^2 \right]_0^2 dz = \int_0^1 \left[\tfrac{3}{4} (2)^2 z^2 - 0 \right] dz$$

$$= \int_0^1 3z^2 \, dz = \frac{3z^3}{3} \Big|_0^1 = \left[z^3 \right]_0^1 = \underline{\underline{1}}$$

2) Evaluate $\int_0^a \int_0^a \int_0^a (x^2+y^2+z^2)\,dx\,dy\,dz$

Solution: $\int_0^a \int_0^a \int_0^a (x^2+y^2+z^2)\,dx\,dy\,dz$

$= \int_0^a \int_0^a \left[x^3/3 + xy^2 + xz^2 \right]_0^a dy\,dz$

$= \int_0^a \int_0^a \left[a^3/3 + ay^2 + az^2 - 0 \right] dy\,dz$

$= \int_0^a \int_0^a \left[\frac{a^3}{3} + ay^2 + az^2 \right] dy\,dz$

$= \int_0^a \left[a^3/3\, y + a\, y^3/3 + az^2 y \right]_0^a dz$

$= \int_0^a \left[\frac{a^4}{3} + \frac{a^4}{3} + a^2z^2 \right] dz$

$= \int_0^a \left[\frac{2}{3}a^4 + a^2z^2 \right] dz = \left[\frac{2}{3}a^4 z + a^2 \frac{z^3}{3} \right]_0^a$

$= \left[\frac{2}{3}a^5 + \frac{a^5}{3} \right] = \frac{3a^5}{3} = a^5$//

3) Evaluate $\displaystyle\int_0^4 \int_0^{2\sqrt{z}} \int_0^{\sqrt{4z-x^2}} dy\, dx\, dz$

<u>Solution</u> : $\displaystyle\int_0^4 \int_0^{2\sqrt{z}} \int_0^{\sqrt{4z-x^2}} dy\, dx\, dz$

$$= \int_0^4 \left\{ \int_0^{2\sqrt{z}} \left[\int_0^{\sqrt{4z-x^2}} dy \right] dx \right\} dz$$

$$= \int_0^4 \left\{ \int_0^{2\sqrt{z}} y \Big]_0^{\sqrt{4z-x^2}} dx \right\} dz$$

$$= \int_0^4 \left\{ \int_0^{2\sqrt{z}} \left[\sqrt{4z-x^2} \right] dx \right\} dz$$

$$= \int_0^4 \left[\frac{x}{2} \sqrt{4z-x^2} + \frac{4z}{2} \sin^{-1} \frac{x}{2\sqrt{z}} \right]_0^{2\sqrt{z}} dz$$

put: $x = 2\sqrt{z}$

$x^2 = 4z$

$$= \int_0^4 \left[\frac{\cancel{2}\sqrt{z}}{\cancel{2}} \cancel{\sqrt{4z-4z}}^{\,0} + \frac{\cancel{4}^{2} z}{\cancel{2}_{1}} \sin^{-1}\left(\frac{\cancel{2\sqrt{z}}}{\cancel{2\sqrt{z}}}\right) \right] dz$$

$$= \int_0^4 2z \sin^{-1}(1)\, dz \quad = \int_0^4 \cancel{2} z \, \frac{\pi}{\cancel{2}} dz$$

$$= \int_0^4 \pi z \, dz$$

$$= \pi \left[\frac{z^2}{2} \right]_0^4 = \pi \left[16/2 - 0 \right] = \underline{\underline{8\pi}}$$

4) $\displaystyle\int_{-1}^{1} \int_{0}^{z} \int_{x-z}^{x+z} (x+y+z) \, dy \, dx \, dz$

Solution: $\displaystyle\int_{-1}^{1} \int_{0}^{z} \int_{x-z}^{x+z} (x+y+z) \, dy \, dx \, dz$

$$= \int_{-1}^{1} \int_{0}^{z} \left[xy + y^2/2 + zy \right]_{x-z}^{x+z} dx \, dz$$

$$= \int_{-1}^{1} \int_{0}^{z} \left[x(x+z) + \frac{(x+z)^2}{2} + z(x+z) - x(x-z) - \frac{(x-z)^2}{2} - z(x-z) \right] dx \, dz$$

$$= \int_{-1}^{1} \int_{0}^{z} \left[x^2 + xz + \frac{x^2+2xz+z^2}{2} + zx + z^2 - x^2 + xz - \frac{x^2+2xz-z^2}{2} - zx + z^2 \right] dx \, dz$$

$$= \int_{-1}^{1}\int_{0}^{z} [4xz + 2z^2]\, dx\, dz$$

$$= \int_{-1}^{1} \left[4\frac{x^2}{2}z + 2z^2 x\right]_0^z dz$$

$$= \int_{-1}^{1} \left[2z^3 + 2z^3 - 0\right] dz$$

$$= \int_{-1}^{1} 4z^3\, dz$$

$$= \left[4\frac{z^4}{4}\right]_{-1}^{1} = 1\left[z^4\right]_{-1}^{1} = \cancel{[1+1]} = \cancel{0}$$

$$= [1-1] = 0$$

5) Evaluate $\int_0^a \int_0^x \int_0^{x+y} e^{x+y+z}\, dz\, dy\, dx$

Solution: $\int_0^a \int_0^x \int_0^{x+y} e^{x+y+z}\, dz\, dy\, dx$

$$= \int_0^a \int_0^x \int_0^{x+y} e^x e^y e^z \, dz \, dy \, dx$$

$$= \int_0^a e^x \left\{ \int_0^x e^y \left[\int_0^{x+y} e^z dz \right] dy \right\} dx$$

$$= \int_0^a e^x \left\{ \int_0^x e^y \left[e^z \right]_0^{x+y} dy \right\} dx$$

$$= \int_0^a e^x \left\{ \int_0^x e^y \left[e^{x+y} - e^0 \right] dy \right\} dx$$

$$= \int_0^a e^x \left\{ \int_0^x e^y \left[e^{x+y} - 1 \right] dy \right\} dx$$

$$= \int_0^a e^x \left\{ \int_0^x (e^x . e^{2y} - e^y) \, dy \right\} dx$$

$$= \int_0^a e^x \left[e^x \frac{e^{2y}}{2} - e^y \right]_0^x dx$$

$$= \int_0^a e^x \left[\left(e^x \frac{e^{2x}}{2} - e^x \right) - \left(e^x \frac{e^0}{2} - e^0 \right) \right] dx$$

$$= \int_0^a e^x \left[\frac{e^{3x}}{2} - e^x - \frac{e^x}{2} + 1 \right] dx$$

$$= \int_0^a e^x \left[\frac{e^{3x}}{2} - \frac{3}{2}e^x + 1 \right] dx$$

$$= \int_0^a \left[\frac{e^{4x}}{2} - \frac{3}{2}e^{2x} + e^x \right] dx$$

$$= \left[\frac{e^{4x}}{8} - \frac{3e^{2x}}{4} + e^x \right]_0^a$$

$$= \left[\frac{e^{4a}}{8} - \frac{3}{4}e^{2a} + e^a - \left(\frac{e^{4(0)}}{8} - \frac{3}{4}e^{2(0)} + e^0 \right) \right]$$

$$= \left[\frac{e^{4a}}{8} - \frac{3}{4}e^{2a} + e^a - \frac{1}{8} + 3/4 - 1 \right]$$

$$= \frac{e^{4a}}{8} - \frac{3}{4}e^{2a} + e^a - 3/8$$